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Application of Fractional Fourier series in Evaluating Fractional Integrals

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Abstract: **In this paper, we use fractional Fourier series to solve three types of fractional integrals based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus. Fractional Euler's formula, fractional DeMoivre's formula and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our results are generalization of the traditional calculus results. On the other hand, three examples are given to illustrate our results.**

Keyword: **fractional Fourier series, fractional integrals, fractional Euler's formula, fractional DeMoivre's formula, new multiplication, fractional analytic functions.**

I. INTRODUCTION

In recent years, fractional calculus has attracted more and more attention due to its wide application in science and engineering [1-11]. However, the definition of fractional derivative is not unique, there are many useful definitions, including Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [12-15]. Jumarie modified the definition of R-L fractional derivative with a new formula, and we obtained that the modified fractional derivative of a constant function is zero. Thus, it is easier to connect fractional calculus with traditional calculus by using this definition.

In this paper, based on Jumarie type of R-L fractional calculus, we use the fractional Fourier series to solve the following three types of α-fractional integrals:

$$\left({}_{0}I^{\alpha}_{T_\alpha}\right)\begin{bmatrix}(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]+r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\\ \otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes-1}\end{bmatrix}, \quad (1)$$

$$\left({}_{0}I^{\alpha}_{T_\alpha}\right)\begin{bmatrix}(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]+r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\\ \otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes-1}\otimes cos_\alpha(kt^\alpha)\end{bmatrix}, \quad (2)$$

and

$$\left({}_{0}I^{\alpha}_{T_\alpha}\right)\begin{bmatrix}-(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]+r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]\\ \otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes-1}\otimes sin_\alpha(kt^\alpha)\end{bmatrix}. \quad (3)$$

Where $0<\alpha\leq 1$, a,b,c,r are real numbers, k is any positive integer, and $a\neq 0, b\neq 0, a\neq b, |r|<|a|, |r|<|b|$. The major methods used in this article are fractional Euler's formula, fractional DeMoivre's formula, and a new multiplication of fractional analytic functions. In fact, the results we obtained are natural generalization of classical calculus results. In addition, some examples are provided to illustrate our results.

II. DEFINITIONS AND PROPERTIES

First, we introduce fractional calculus and some important properties used in this paper.

Definition 2.1 ([16])**:** If $0 < \alpha \leq 1$, and t_0 is a real number. The Jumarie's modified R-L α-fractional derivative is defined by

$$\left({}_{t_0}D_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t_0}^{t}\frac{f(x)-f(t_0)}{(t-x)^{\alpha}}dx, \tag{4}$$

And the Jumarie's modified R-L α-fractional integral is defined by

$$\left({}_{t_0}I_t^{\alpha}\right)[f(t)] = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(x)}{(t-x)^{1-\alpha}}dx\ , \tag{5}$$

where $\Gamma(w) = \int_0^{\infty} s^{w-1}e^{-s}ds$ is the gamma function defined on $w > 0$.

Proposition 2.2 ([17])**:** *If α, β, t_0, C are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{t_0}D_t^{\alpha}\right)\left[(t-t_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha}, \tag{6}$$

and

$$\left({}_{t_0}D_t^{\alpha}\right)[C] = 0. \tag{7}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([18])**:** If t, t_0, and a_k are real numbers for all k, $t_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_{\alpha}: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_{\alpha}(t^{\alpha}) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha}$ on some open interval containing t_0, then we say that $f_{\alpha}(t^{\alpha})$ is α-fractional analytic at x_0. In addition, if $f_{\alpha}: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_{α} is called an α-fractional analytic function on $[a, b]$.

Definition 2.4 ([19])**:** If $0 < \alpha \leq 1$, and t_0 is a real number. Let $f_{\alpha}(t^{\alpha})$ and $g_{\alpha}(t^{\alpha})$ be two α-fractional analytic functions defined on an interval containing t_0 ,

$$f_{\alpha}(t^{\alpha}) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}, \tag{8}$$

$$g_{\alpha}(t^{\alpha}) = \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \tag{9}$$

Then

$$\begin{aligned} & f_{\alpha}(t^{\alpha}) \otimes g_{\alpha}(t^{\alpha}) \\ & = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} \otimes \sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)(t-t_0)^{k\alpha}. \end{aligned} \tag{10}$$

Equivalently,

$$\begin{aligned} & f_{\alpha}(t^{\alpha}) \otimes g_{\alpha}(t^{\alpha}) \\ & = \sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty}\frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k} \\ & = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m}a_{k-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{11}$$

Definition 2.5 ([19])**:** If $0 < \alpha \leq 1$, and t is any real number. The α-fractional exponential function is defined by

$$E_{\alpha}(t^{\alpha}) = \sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes k}. \tag{12}$$

In addition, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes 2k}, \tag{13}$$

and

$$sin_\alpha(t^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}t^\alpha\right)^{\otimes(2k+1)}. \tag{14}$$

Definition 2.6 ([20])**:** If $0 < \alpha \le 1$, and r is any real number. Let $f_\alpha(t^\alpha)$ be a α-fractional analytic function. Then the α-fractional analytic function $f_\alpha(t^\alpha)^{\otimes r}$ is defined by

$$f_\alpha(t^\alpha)^{\otimes r} = E_\alpha\left(r \cdot Ln_\alpha\left(f_\alpha(t^\alpha)\right)\right). \tag{15}$$

Definition 2.7 ([21])**:** Suppose that $0 < \alpha \le 1$, $i = \sqrt{-1}$, and $f_\alpha(t^\alpha)$, $g_\alpha(t^\alpha)$, $p_\alpha(t^\alpha)$, $q_\alpha(t^\alpha)$ are α-fractional real analytic at $t = t_0$. Let $z_\alpha(t^\alpha) = f_\alpha(t^\alpha) + ig_\alpha(t^\alpha)$ and $w_\alpha(t^\alpha) = p_\alpha(t^\alpha) + iq_\alpha(t^\alpha)$ be complex analytic at $t = t_0$. Define

$$\begin{aligned} & z_\alpha(t^\alpha) \otimes w_\alpha(t^\alpha) \\ &= \left(f_\alpha(t^\alpha) + i\, g_\alpha(t^\alpha)\right) \otimes \left(p_\alpha(t^\alpha) + i\, q_\alpha(t^\alpha)\right) \\ &= [f_\alpha(t^\alpha) \otimes p_\alpha(t^\alpha) - g_\alpha(t^\alpha) \otimes q_\alpha(t^\alpha)] + i[f_\alpha(t^\alpha) \otimes q_\alpha(t^\alpha) + g_\alpha(t^\alpha) \otimes p_\alpha(t^\alpha)]. \end{aligned} \tag{16}$$

In addition, we define

$$|\, z_\alpha(t^\alpha)|_\otimes = ([f_\alpha(t^\alpha)]^{\otimes 2} + [g_\alpha(t^\alpha)]^{\otimes 2})^{\otimes \frac{1}{2}}. \tag{17}$$

Proposition 2.8 (fractional Euler's formula) ([22])**:** *Let* $0 < \alpha \le 1$, t *be a real number, then*

$$E_\alpha(it^\alpha) = cos_\alpha(t^\alpha) + isin_\alpha(t^\alpha). \tag{18}$$

Proposition 2.9 (fractional DeMoivre's formula) ([22])**:** *Let* $0 < \alpha \le 1$, n *be any positive integer, then*

$$\left(cos_\alpha(t^\alpha) + isin_\alpha(t^\alpha)\right)^{\otimes n} = cos_\alpha(nt^\alpha) + isin_\alpha(nt^\alpha). \tag{19}$$

Definition 2.10 (fractional Fourier series) ([21])**:** If $0 < \alpha \le 1$, and $f_\alpha(t^\alpha)$ is α-fractional analytic at $t = 0$ with the same period T_α of $E_\alpha(it^\alpha)$. Then the α-fractional Fourier series expansion of $f_\alpha(t^\alpha)$ is

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k cos_\alpha(kt^\alpha) + b_k sin_\alpha(kt^\alpha), \tag{20}$$

where
$$\begin{cases} a_0 = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^\alpha\right)[f_\alpha(t^\alpha)], \\ a_k = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^\alpha\right)[f_\alpha(t^\alpha) \otimes cos_\alpha(kt^\alpha)], \\ b_k = \frac{2}{T_\alpha}\left({}_0I_{T_\alpha}^\alpha\right)[f_\alpha(t^\alpha) \otimes sin_\alpha(kt^\alpha)], \end{cases} \tag{21}$$

for all positive integers k.

III. RESULTS AND EXAMPLES

To obtain the major results in this paper, the following lemmas are needed.

Lemma 3.1 (fractional geometric series)**:** *If* $0 < \alpha \le 1$ *and let* $z_\alpha(t^\alpha)$ *be a complex* α*-fractional analytic function such that* $|z_\alpha(t^\alpha)|_\otimes < 1$. *Then*

$$[1 + z_\alpha(t^\alpha)]^{\otimes -1} = \sum_{k=0}^{\infty}(-1)^k\, [z_\alpha(t^\alpha)]^{\otimes k}. \tag{22}$$

Proof Since $|\, z_\alpha(t^\alpha)|_\otimes < 1$, it follows that

$$\begin{aligned} & [1 + z_\alpha(t^\alpha)] \otimes \sum_{k=0}^{\infty}(-1)^k\, [z_\alpha(t^\alpha)]^{\otimes k} \\ &= [1 + z_\alpha(t^\alpha)] \otimes [1 - z_\alpha(t^\alpha) + [z_\alpha(t^\alpha)]^{\otimes 2} - [z_\alpha(t^\alpha)]^{\otimes 3} + \cdots] \\ &= 1. \end{aligned}$$

And hence, the desired result holds. Q.e.d.

Lemma 3.2: *If* $0 < \alpha \leq 1$, a, b, c *are real numbers and* $a \neq 0, b \neq 0$, $a \neq b$. *Let* $z_\alpha(t^\alpha)$ *be a complex* α*-fractional analytic function such that* $|z_\alpha(t^\alpha)|_\otimes < |a|$, $|z_\alpha(t^\alpha)|_\otimes < |b|$. *Then*

$$(z_\alpha(t^\alpha)+c)\otimes\left[\left(z_\alpha(t^\alpha)\right)^{\otimes 2}+(a+b)\cdot z_\alpha(t^\alpha)+ab\right]^{\otimes -1}=\frac{c}{ab}+\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)[z_\alpha(t^\alpha)]^{\otimes k}. \quad (23)$$

Proof $\quad (z_\alpha(t^\alpha)+c)\otimes\left[\left(z_\alpha(t^\alpha)\right)^{\otimes 2}+(a+b)\cdot z_\alpha(t^\alpha)+ab\right]^{\otimes -1}$

$$=\frac{a-c}{a-b}[z_\alpha(t^\alpha)+a]^{\otimes -1}+\frac{c-b}{a-b}[z_\alpha(t^\alpha)+b]^{\otimes -1}$$

$$=\frac{a-c}{a(a-b)}\left[1+\frac{1}{a}\cdot z_\alpha(t^\alpha)\right]^{\otimes -1}+\frac{c-b}{b(a-b)}\left[1+\frac{1}{b}\cdot z_\alpha(t^\alpha)\right]^{\otimes -1}$$

$$=\frac{a-c}{a(a-b)}\sum_{k=0}^{\infty}\frac{(-1)^k}{a^k}[z_\alpha(t^\alpha)]^{\otimes k}+\frac{c-b}{b(a-b)}\sum_{k=0}^{\infty}\frac{(-1)^k}{b^k}[z_\alpha(t^\alpha)]^{\otimes k} \text{ (by fractional geometric series)}$$

$$=\frac{c}{ab}+\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)[z_\alpha(t^\alpha)]^{\otimes k}.$$

Q.e.d.

Lemma 3.3: *If* $0 < \alpha \leq 1$, a, b, c, r *are real numbers and* $a \neq 0, b \neq 0$, $a \neq b$, $|r| < |a|$, $|r| < |b|$. *Then*

$$\begin{aligned}&(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]\\&\qquad +r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\end{aligned}$$

$$\otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes -1}.$$

$$=\frac{c}{ab}+\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)r^k\cdot cos_\alpha(kt^\alpha). \quad (24)$$

And

$$\begin{aligned}&-(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\\&\qquad +r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]\end{aligned}$$

$$\otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes -1}$$

$$=\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)r^k\cdot sin_\alpha(kt^\alpha). \quad (25)$$

Proof In Lemma 3.2, let $z_\alpha(t^\alpha)=r\cdot E_\alpha(it^\alpha)$, then by fractional Euler's formula, we know that

$$z_\alpha(t^\alpha)=r\cdot cos_\alpha(t^\alpha)+ir\cdot sin_\alpha(t^\alpha). \quad (26)$$

Using Lemma 3.2 yields

$$(r\cdot cos_\alpha(t^\alpha)+ir\cdot sin_\alpha(t^\alpha)+c)\otimes\left[\left(r\cdot cos_\alpha(t^\alpha)+ir\cdot sin_\alpha(t^\alpha)\right)^{\otimes 2}+(a+b)\cdot\left(r\cdot cos_\alpha(t^\alpha)+ir\cdot sin_\alpha(t^\alpha)\right)+ab\right]^{\otimes -1}$$

$$=\frac{c}{ab}+\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)[r\cdot cos_\alpha(t^\alpha)+ir\cdot sin_\alpha(t^\alpha)]^{\otimes k}. \quad (27)$$

Thus, by fractional De Moivre's formula

$$\begin{aligned}&(r\cdot cos_\alpha(t^\alpha)+c\\&\qquad +ir\cdot sin_\alpha(t^\alpha))\otimes\left[r^2\cdot cos_\alpha(2t^\alpha)+ir^2\cdot sin_\alpha(2t^\alpha)+(a+b)\cdot\left(r\cdot cos_\alpha(t^\alpha)+ir\cdot sin_\alpha(t^\alpha)\right)\right.\\&\qquad \left.+ab\right]^{\otimes -1}\end{aligned}$$

$$=\frac{c}{ab}+\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)[r^k\cdot cos_\alpha(kt^\alpha)+ir^k\cdot sin_\alpha(kt^\alpha)]. \quad (28)$$

Therefore,

$$\begin{aligned}&(r\cdot cos_\alpha(t^\alpha)+c+ir\cdot sin_\alpha(t^\alpha))\otimes\big[[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]\\&\qquad\qquad+i[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\big]^{\otimes-1}\\&=\frac{c}{ab}+\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)[r^k\cdot cos_\alpha(kt^\alpha)+ir^k\cdot sin_\alpha(kt^\alpha)].\end{aligned} \tag{29}$$

Hence,

$$\begin{aligned}&(r\cdot cos_\alpha(t^\alpha)+c+ir\cdot sin_\alpha(t^\alpha))\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]\\&\qquad\qquad-i[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\\&\otimes[[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]^{\otimes2}+[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]^{\otimes2}]^{\otimes-1}\\&=\frac{c}{ab}+\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)[r^k\cdot cos_\alpha(kt^\alpha)+ir^k\cdot sin_\alpha(kt^\alpha)].\end{aligned} \tag{30}$$

Finally, we obtain

$$\begin{aligned}&(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]\\&\qquad\qquad+r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\\&\otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes-1}\\&=\frac{c}{ab}+\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)r^k\cdot cos_\alpha(kt^\alpha).\end{aligned}$$

And

$$\begin{aligned}&-(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\\&\qquad\qquad+r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]\\&\otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes-1}\\&=\frac{1}{a-b}\sum_{k=1}^{\infty}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)r^k\cdot sin_\alpha(kt^\alpha).\end{aligned}$$

Q.e.d.

The following is the major results in this paper, we find the fractional integral of some fractional trigonometric functions.

Theorem 3.4: *Suppose that* $0<\alpha\le1$, a,b,c,r *are real numbers and* $a\neq0, b\neq0, a\neq b$, $|r|<|a|, |r|<|b|$. *Then the* α*-fractional integrals*

$$\left({}_0I^{\alpha}_{T_\alpha}\right)\left[\begin{matrix}(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]+r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\\ \otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes-1}\end{matrix}\right]$$
$$=\frac{c}{ab}\cdot T_\alpha\,. \tag{31}$$

And

$$\left({}_0I^{\alpha}_{T_\alpha}\right)\left[\begin{matrix}(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]+r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]\\ \otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes-1}\otimes cos_\alpha(kt^\alpha)\end{matrix}\right]$$
$$=\frac{1}{a-b}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)r^k\cdot\frac{T_\alpha}{2}\,. \tag{32}$$

$$\left({}_0I^{\alpha}_{T_\alpha}\right)\left[\begin{matrix}-(r\cdot cos_\alpha(t^\alpha)+c)\otimes[r^2\cdot sin_\alpha(2t^\alpha)+(a+b)r\cdot sin_\alpha(t^\alpha)]+r\cdot sin_\alpha(t^\alpha)\otimes[r^2\cdot cos_\alpha(2t^\alpha)+(a+b)r\cdot cos_\alpha(t^\alpha)+ab]\\ \otimes[2abr^2\cdot cos_\alpha(2t^\alpha)+[2(a+b)r^3+2ab(a+b)r]\cdot cos_\alpha(t^\alpha)+r^4+(a+b)^2r^2+a^2b^2]^{\otimes-1}\otimes sin_\alpha(kt^\alpha)\end{matrix}\right]$$
$$=\frac{1}{a-b}(-1)^k\left(\frac{a-c}{a^{k+1}}+\frac{c-b}{b^{k+1}}\right)r^k\cdot\frac{T_\alpha}{2}\,. \tag{33}$$

for all positive integers k.

Proof By the definition of fractional Fourier series and Lemma 3.3, the desired results hold. Q.e.d.

Finally, some examples are provided to illustrate our results.

Example 3.5: If $0 < \alpha \le 1$, and let $r = 2, a = -3, b = 4, c = -1$, then by Theorem 3.4, we obtain the α-fractional integral

$$\left({}_0I^{\alpha}_{T_\alpha}\right)\left[\begin{matrix}(2\cdot cos_\alpha(t^\alpha)-1)\otimes[4\cdot cos_\alpha(2t^\alpha)+2\cdot cos_\alpha(t^\alpha)-12]+2\cdot sin_\alpha(t^\alpha)\otimes[4\cdot sin_\alpha(2t^\alpha)+2\cdot sin_\alpha(t^\alpha)]\\ \otimes[-96\cdot cos_\alpha(2t^\alpha)-32\cdot cos_\alpha(t^\alpha)+164]^{\otimes -1}\end{matrix}\right]=\frac{1}{12}\cdot T_\alpha\,. \tag{34}$$

Example 3.6: Let $0 < \alpha \le 1$, and let $r = \frac{1}{3}, a = 2, b = 1, c = -3, k = 5$, then by Theorem 3.4, the α-fractional integral

$$\left({}_0I^{\alpha}_{T_\alpha}\right)\left[\begin{matrix}\left(\frac{1}{3}\cdot cos_\alpha(t^\alpha)-3\right)\otimes\left[\frac{1}{9}\cdot cos_\alpha(2t^\alpha)+cos_\alpha(t^\alpha)+2\right]+\frac{1}{3}\cdot sin_\alpha(t^\alpha)\otimes\left[\frac{1}{9}\cdot sin_\alpha(2t^\alpha)+sin_\alpha(t^\alpha)\right]\\ \otimes\left[\frac{4}{9}\cdot cos_\alpha(2t^\alpha)+\frac{114}{27}\cdot cos_\alpha(t^\alpha)+\frac{406}{81}\right]^{\otimes -1}\otimes cos_\alpha(5t^\alpha)\end{matrix}\right]=\frac{251}{31104}\cdot T_\alpha\,. \tag{35}$$

Example 3.7: If $0 < \alpha \le 1$, and let $r = 1, a = -4, b = 2, c = 3, k = 4$, then using Theorem 3.4 yields the α-fractional integral

$$\left({}_0I^{\alpha}_{T_\alpha}\right)\left[\begin{matrix}-(cos_\alpha(t^\alpha)+3)\otimes[sin_\alpha(2t^\alpha)-2\cdot sin_\alpha(t^\alpha)]+sin_\alpha(t^\alpha)\otimes[cos_\alpha(2t^\alpha)-2\cdot cos_\alpha(t^\alpha)-8]\\ \otimes[-16\cdot cos_\alpha(2t^\alpha)+28\cdot cos_\alpha(t^\alpha)+69]^{\otimes -1}\otimes sin_\alpha(4t^\alpha)\end{matrix}\right]=-\frac{13}{4096}\cdot T_\alpha\,. \tag{36}$$

IV. CONCLUSION

The main purpose of this paper is to use fractional Fourier series to solve some types of fractional integrals. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalization of the results of ordinary calculus. On the other hand, some examples are provided to illustrate our results. In the future, we will continue to use fractional Fourier series method to study the problems in fractional calculus and fractional differential equations.

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